$K$-means: Relaxation and Correction

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HDPA - July 2nd
Clustering arises in various contexts

Clustering individuals w.r.t. features

Clustering features

Clustering graphs
Objectives

Topic of the talk

- $K$-means (relaxed or not) must and can be debiased
- we derive some non-asymptotic partial recovery bounds for a relaxed $K$-means
- some optimality in terms of exponential exponent

Main message

A corrected convex relaxation of $K$-means achieves some rate-optimal performances in various settings including (conditional) mixture of sub-Gaussian and (conditional) Stochastic Block Model.

Only tuning Parameter is $K$
1. Two clustering Models

2. $K$-means and relaxed $K$-means

3. Corrected $K$-means

4. Partial Recovery bounds
   - subGaussian Mixtures
   - Stochastic Block Models
Partition

Partition $G^* = \{G_1^*, \ldots, G_K^*\}$ of $\{1, \ldots, n\}$

Mixture of subGaussian variables (conditional)

$X_1, \ldots, X_n \in \mathbb{R}^p$ are independent with
- $\mathbb{E}[X_a] = \mu_k$ if $a \in G_k^*$
- $\Sigma_a^{-1/2} X_a$ is SubGauss($L^2 I_p$) where $\Sigma_a = \text{Cov}(X_a)$ and $L \geq 1$.

The observations are gathered in $X = [X_1, \ldots, X_n] \in \mathbb{R}^{p \times n}$
Partition $G^* = \{G^*_1, \ldots, G^*_K\}$ of $\{1, \ldots, n\}$

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The observations are gathered in $X = [X_1, \ldots, X_n] \in \mathbb{R}^{p \times n}$

Objective: recovering $G^*$ from $X$ ($\mu$ and $\Sigma$ are unknown but $K$ is known)
Holland et al (83), Abbe (’17),
Let $X$ = adjacency matrix of an undirected graph $\in \{0, 1\}^{n \times n}$.

Let $Q \in [0, 1]_{sym}^{K \times K}$

(conditional) SBM

The graph is generated by a SBM with partition $G^*$ and matrix $Q$ if $X_{ab}$ with $a < b$ are independent and

$$
P[X_{ab} = 1] = Q_{jk} \quad \text{for any } a \in G_j^* \text{ and } b \in G_k^* ,$$
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**Objective**: recovering $G^*$ from $X$ ($Q$ is unknown.)
1 Two clustering Models

2 $K$-means and relaxed $K$-means

3 Corrected $K$-means

4 Partial Recovery bounds
   - subGaussian Mixtures
   - Stochastic Block Models
How do we encode partition learning?

Membership Matrix $A \in \{0, 1\}^{n \times K}$ defined by $A_{ak} = 1_{a \in G_k}$ (or equivalently function $k : [n] \mapsto [K]$)
is NOT Identifiable. Why?
How do we encode partition learning?

**Membership Matrix** $A \in \{0, 1\}^{n \times K}$ defined by $A_{ak} = 1_{a \in G_k}$ (or equivalently function $k : [n] \mapsto [K]$)
is at best identifiable up to permutation

A more suitable object: The $n \times n$ **Partnership matrix**

$$B^* = A(A^TA)^{-1}A^T$$

$$B^*_{ab} = \begin{cases} 
\frac{1}{|G^*_k|} & \text{if } a \text{ and } b \text{ belong to the same } G^*_k \\
0 & \text{else} 
\end{cases}$$

Invariant with respect to the group labeling.
\[ \hat{G} \in \arg \min_{G} \text{Crit}(X, G) \text{ where} \]

\[ \text{Crit}(X, G) = \sum_{k=1}^{K} \sum_{a \in G_k} \| X_a - \bar{X}_{G_k} \|^{2}, \]

where \( \bar{X}_{G_k} = \frac{1}{|G_k|} \sum_{a \in G_k} X_a \)

In practice, iterative minimization based on Lloyd’s algorithm LLoyd(’82).
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Two steps:

1. Compute the centroids
2. Update the partition
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Two steps:

1. Compute the centroids
2. Update the partition
**K-means criterion**

\[ \hat{G} \in \arg \min_G \text{Crit}(X, G) \]

where

\[ \text{Crit}(X, G) = \sum_{k=1}^{K} \sum_{a \in G_k} \|X_a - \overline{X}_{G_k}\|^2, \]

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In practice, iterative minimization based on Lloyd’s algorithm \( \text{LLoyd}(’82) \).

Two steps:

1. Compute the centroids
2. Update the partition

Two caveats:

- There can be many local optima.
- In worst-case solving K-means is \( NP \)-hard (Mahajan et al. ’09)
Rewriting $K$-means

\[
\text{Crit}(X, G) = \sum_k |G_k| \|X G_k\|^2 - 2 \sum_{a,b \in G_k} \langle X_a, X_b \rangle \frac{1}{|G_k|} + \sum_a \|X_a\|^2
\]

\[
= -\sum_k \sum_{a,b \in G_k} \langle X_a, X_b \rangle \frac{1}{|G_k|} + \ldots
\]

\[
= -\langle X^T X, B \rangle + \ldots
\]
Rewriting $K$-means

$$\text{Crit}(X, G) = \sum_k |G_k| \|\bar{X}G_k\|^2 - 2 \sum_{a,b \in G_k} \langle X_a, X_b \rangle \frac{1}{|G_k|} + \sum_a \|X_a\|^2$$

$$= -\sum_k \sum_{a,b \in G_k} \langle X_a, X_b \rangle \frac{1}{|G_k|} + \ldots$$

$$= -\langle X^T X, B \rangle + \ldots$$

**Lemma** *(Peng & Wei (07))*

The $K$-means minimizer $\hat{G}$ satisfies

$$\hat{B} \in \arg \min_{B \in \mathcal{D}} \langle -X^T X, B \rangle,$$

$$\mathcal{D} := \left\{ B \in \mathbb{R}^{p \times p} : \begin{array}{l}
\bullet \ B \succeq 0 \\
\bullet \ \sum_a B_{ab} = 1, \forall b \\
\bullet \ B_{ab} \geq 0, \forall a, b \\
\bullet \ \text{Tr}(B) = K \\
\bullet \ B^2 = B \end{array} \right\}$$

**Proof**: Perron-Frobenius
Relaxed $K$-means


1 Estimate $B^*$ using the semi-definite program (SDP)

$$\hat{B} = \arg\min_{B \in \mathcal{C}} \langle -X^TX, B \rangle$$

where

$$\mathcal{C} := \left\{ B \in \mathbb{R}^{n \times n} : \begin{array}{l} \bullet B \succcurlyeq 0 \\ \bullet \sum_a B_{ab} = 1, \forall b \\ \bullet B_{ab} \geq 0, \forall a, b \\ \bullet \text{Tr}(B) = K \end{array} \right\}$$

2 (Compute $\hat{G}$ by applying any clustering algorithm on $\hat{B}$)

Remark:

- Convex optimization but many constraints:
  
  [https://cims.nyu.edu/~villar/mnist.html](https://cims.nyu.edu/~villar/mnist.html)

- No information of the group sizes are needed.
A second relaxation: Spectral Clustering

Spectral Clustering

1. Compute the matrix $\hat{U}$ made of the $K$-leading eigenvectors of $X^TX$.
2. Estimate $\hat{G}$ by distance clustering on the rows of $\hat{U}$.

(e.g. Apply an approximate $K$-means algorithm to the rows of the matrix $\hat{U}$)
A second relaxation: Spectral Clustering

### Spectral Clustering

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---

### Lemma (Peng & Wei(07))

Spectral Clustering is equivalent to

1. Estimate $B^*$ using the semi-definite program (SDP)

$$
\overline{B} = \arg\min_{B \in \overline{C}} \langle -X^TX, B \rangle
$$

$$
\overline{C} := \left\{ B \in \mathbb{R}^{p \times p} : \begin{array}{l}
1 \succ B \succ 0 \\
\text{Tr}(B) = K
\end{array} \right\}
$$

2. Compute $\hat{G}$ by distance clustering on the rows of $\overline{B}$

$\Rightarrow$ it amounts to dropping the constraints $B1 = 1, B_{ab} \geq 0$ in the former relaxation

**Proof**: 1) $\overline{B} = \hat{U}\hat{U}^T$
2) $(\hat{U}\hat{U}^T)_{a\bullet}$ is some orthogonal transformation of $\hat{U}_{a\bullet}$. 
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Wikipedia knows

$$\text{Crit}_K(G) = \sum_{k=1}^{K} \sum_{a \in G_k} \|X_a - \overline{X_{G_k}}\|^2$$

Quantization rather than clustering

Caveat

A simple model

Assume that the "points" $X_a$ are independent random variables with

$$
\mathbb{E}[X_a] = \mu_a \quad \text{and} \quad \text{Tr}(\text{Cov}(X_a)) = \Gamma_a.
$$

$$
\text{Crit}_K(G) = \sum_{k=1}^{K} \sum_{a \in G_k} \|X_a - \overline{X}_{G_k}\|^2
$$

Expected value at $G$

For a partition $G$ we have

$$
\mathbb{E}[\text{Crit}_K(G)] = \frac{1}{2} \sum_{k} \frac{1}{|G_k|} \sum_{a,b \in G_k} \|\mu_a - \mu_b\|^2 + \sum_{a} \Gamma_a - \sum_{k} \frac{1}{|G_k|} \sum_{a \in G_k} \Gamma_a
$$

$\rightarrow$ tends to split "wide" clusters: a correction is needed!
Recall our Minimization Problem: $\langle -X^T X, B \rangle$

sGaussian Mixtures are of the form: $X_a = \mathbb{E}[X_a] + E_a = Information + Noise$,

$$\mathbb{E}[X^T X] = \mathbb{E}[X]^T \mathbb{E}[X] + \Gamma,$$

where $\Gamma_{aa} = \text{Tr}[\text{Cov}(E_a)]$

Population $K$-means vs Ideal $K$-means

$$B^{\text{pop}} = \arg \min_{B \in \mathcal{D}} \langle -\mathbb{E}[X]^T \mathbb{E}[X] - \Gamma, B \rangle$$

$$B^{\text{id}} = \arg \min_{B \in \mathcal{D}} \langle -\mathbb{E}[X]^T \mathbb{E}[X], B \rangle$$

- Since $\text{Tr}[B] = K$, we have $B^{\text{pop}} = B^{\text{id}}$ when $\Gamma = \gamma I$.
- For heterogeneous $\Gamma$, $B^{\text{pop}}_{aa}$ tends to take large values for large $\Gamma_{aa}$ (it splits wide clusters).
Remark: If we knew the groups, we could estimate \( \Gamma = \text{diag}(\Gamma_1, \ldots, \Gamma_n) \) by

\[
\hat{\Gamma}_{aa} = \langle X_a - X_{ne_1(a)}, X_a - X_{ne_2(a)} \rangle
\]

with \( ne_1(a) \) and \( ne_2(a) \) two "neighbors" of \( a \).
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Definition

Then, the estimator \( \hat{\Gamma} \) is the diagonal matrix defined by

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Definition

Set $U(a, b) := \max_{c, d \in [n]\{a, b\}} \left| \langle X_a - X_b, \frac{X_c - X_d}{\|X_c - X_d\|} \rangle \right|$ and

$$\hat{ne}_1(a) := \arg \min_{b \in [n]\{a\}} U(a, b) \quad \text{and} \quad \hat{ne}_2(a) := \arg \min_{b \in [n]\{a, \hat{ne}_1(a)\}} U(a, b)$$

Then, the estimator $\hat{\Gamma}$ is the diagonal matrix defined by

$$\hat{\Gamma}_{aa} = \langle X_a - X_{\hat{ne}_1(a)}, X_a - X_{\hat{ne}_2(a)} \rangle$$
Corrected relaxed $K$-means

Corrected relaxed $K$-means (Bunea et al. ('16))

Solve the SDP

$$\hat{B} \in \arg\min_{B \in C} \langle -X^T X, B \rangle,$$

with

$$C := \left\{ B \in \mathbb{R}^{n \times n} : \begin{array}{l}
\bullet B \succ 0 \\
\bullet \sum_a B_{ab} = 1, \forall b \\
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\end{array} \right\}$$

Similarly, one may define a corrected spectral clustering.
Corrected relaxed $K$-means

Corrected relaxed $K$-means (Bunea et al. ('16))

Solve the SDP

$$\hat{B} \in \arg\min_{B \in C} \langle \hat{\Gamma} - X^T X, B \rangle,$$

with

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Similarly, one may define a corrected spectral clustering.
Two clustering Models

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Partial recovery bounds

Proportion of misclustered points

\[ err(\hat{G}, G^*) = \min_{\pi \in S_K} \frac{1}{2n} \sum_{k=1}^{K} |G^*_k \triangle \hat{G}_{\pi(k)}| \]

Our goal

Prove that with high-probability, when \( s^2 \) is large

\[ \text{prop. misclustered} = err(\hat{G}, G^*) \leq e^{-cs^2} \]

where \( s^2 \) is an appropriate SNR.

Other related goals:

- **Partial recovery**: Find the minimal \( s^2 \) such that \( err(\hat{G}, G^*) \) is smaller than random guess whp.
- **Perfect recovery**: Find the minimal \( s^2 \) such that \( err(\hat{G}, G^*) = 0 \) whp.
Mixture of subGaussian variables

\( X_1, \ldots, X_n \in \mathbb{R}^p \) are independent with

- \( \mathbb{E}[X_a] = \mu_k \) if \( a \in G^*_k \)
- \( \Sigma^{-1/2}_a X_a \) is SubGauss\((L^2I_p)\) where \( \Sigma_a = \text{cov}(X_a) \)

We set

\[
\Delta^2 = \min_{j \neq k} \| \mu_k - \mu_j \|^2, \quad \sigma^2 = L^2 \max_k |\Sigma_k|_{op} \quad \text{and} \quad R_{\Sigma} = \max_k \frac{|\Sigma_k|^2_F}{|\Sigma_k|_{op}^2},
\]

and define the SNR

\[
s^2 = \frac{\Delta^2}{\sigma^2} \wedge \frac{m \Delta^4}{R_{\Sigma} \sigma^4},
\]

where \( m \) denotes the size of the smallest cluster.
Simplification: $K = 2$, $|G_1^*| = |G_{-1}^*| = n/2$, $\Sigma_1 = \Sigma_{-1} = \sigma^2 I_p$, $\mu_{-1} = -\mu_1$.

Simplified Model 1: $\mu_1$ is known. Bayes Classifier achieves:

$$\mathbb{E}[err(\hat{G}, G^*)] = 2 \mathbb{P}[\mathcal{N}(0, \sigma^2) > ||\mu_1||] \leq 2 \exp \left[ -\frac{\Delta^2}{8\sigma^2} \right]$$

Simplified Model 2: $\mu_1$ is sampled uniformly on the sphere of radius $\Delta/2$. Labels $Z_a \in \{-1, 1\}$ for $a = 1, \ldots, n$ are known.

Objective: classify a new observation $X$.

Optimal Classifier: $\hat{h}(x) = \text{sign} \left( \langle \frac{1}{n} \sum_{a=1}^{n} Z_a X_a, x \rangle \right)$:

- achieves the rate $e^{-c\Delta^2/\sigma^2}$ if $\frac{\Delta^2}{\sigma^2} \gtrsim 1 \vee \frac{p}{n}$.
- achieves the rate $e^{-cn\Delta^4/(p\sigma^4)}$ if $1 \vee \sqrt{\frac{p}{n}} \lesssim \frac{\Delta^2}{\sigma^2} \lesssim 1 \vee \frac{p}{n}$.

See Ndaoud(’18) for proper lower bounds.
Partial recovery bounds

$$s^2 = \frac{\Delta^2}{\sigma^2} \land \frac{m \Delta^4}{R \Sigma \sigma^4},$$

Theorem (Giraud and V. ('18))

If $s^2 \gtrsim n/m$ (+ mild assumption), then $\Pr \left[ \text{err}(\hat{G}, G^*) > e^{-c s^2} \right] \lesssim \frac{1}{n^2}.$
Partial recovery bounds

\[ s^2 = \frac{\Delta^2}{\sigma^2} \land \frac{m\Delta^4}{R\Sigma\sigma^4}, \]

**Theorem (Giraud and V. ('18))**

If \( s^2 \gtrsim n/m \) (+ mild assumption), then \( \mathbb{P} \left[ \text{err}(\hat{G}, G^*) > e^{-cs^2} \right] \lesssim \frac{1}{n^2}. \)

\[ s^2 \gtrsim n/m \] is equivalent to \( \Delta^2 \gtrsim \sigma^2 \frac{n}{m} \left( 1 \lor \sqrt{\frac{R\Sigma}{n}} \right) = \sigma^2 K \left( 1 \lor \sqrt{\frac{R\Sigma}{n}} \right). \)

**Remarks:**

1. \( s^2 \) reduces to \( \Delta^2 / \sigma^2 \) when \( \Delta^2 / \sigma^2 \geq R\Sigma / m \)

Fei and Chen ('18), See Lu and Zhou ('16), Ndaoud('18) for sharp constants
Partial recovery bounds

\[ s^2 = \frac{\Delta^2}{\sigma^2} \wedge \frac{m\Delta^4}{R\Sigma \sigma^4}, \]

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If \( s^2 \gtrsim n/m \) (+ mild assumption), then \( \mathbb{P} \left[ \text{err}(\hat{G}, G^*) > e^{-cs^2} \right] \lesssim \frac{1}{n^2}. \)

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**Remarks:**

1. \( s^2 \) reduces to \( \Delta^2 / \sigma^2 \) when \( \Delta^2 / \sigma^2 \geq R\Sigma / m \)
   Fei and Chen (’18), See Lu and Zhou (’16), Ndaoud(’18) for sharp constants

2. perfect recovery for \( s^2 \gtrsim \log(n) \vee (n/m) = \log(n) \vee K \)
   Dependency in \( K \) is suboptimal.
   Vempala and Wang(’04) \( \rightsquigarrow s^2 \gtrsim \log(n) \vee \sqrt{K \log(n)} \) when \( n \gg p^3. \)

3. Do not cover the case where the proportion of error is \( \geq e^{-c''K} \).
Mild price for \( \Gamma \) estimation:

\[
\frac{\| \Sigma_k \|_{op} \text{Tr}(\Sigma_k)}{\| \Sigma_k \|_F^2} \lesssim \frac{n}{\log(n)}
\]
Benefit of Corrected $K$-means

Mild price for $\Gamma$ estimation:
\[
\frac{\|\Sigma_k\|_{op} \text{Tr}(\Sigma_k)}{\|\Sigma_k\|_F^2} \lesssim \frac{n}{\log(n)}
\]

Without correction, additional assumption is required:
\[
\Delta^2 \gtrsim \frac{\max \Gamma_a - \min \Gamma_a}{m}
\]

For a balanced Partition, it amounts to
\[
\Delta^2 \gtrsim \sigma^2 K \left( 1 + \sqrt{\frac{R\Sigma}{n}} \sqrt{\max_k \text{tr}[\Sigma_k] - \min_k \text{tr}[\Sigma_k]} \right).
\]
**Proof Ideas**

**Simple Versions** : All $\|\mu_i - \mu_j\|_2$ are equal

**Step 1** : $|\hat{\mathbf{B}} - \mathbf{B}^*|_1$ small implies that $\text{err}(\hat{\mathbf{G}}, \mathbf{G})$ is small.

**New Objective** : Show that $\langle \mathbf{X}^T \mathbf{X} - \hat{\Gamma}, \mathbf{B}^* - \mathbf{B} \rangle > 0$ as long as $|\mathbf{B}^* - \mathbf{B}|_1$ is not small
Simple Versions: All $\|\mu_i - \mu_j\|_2$ are equal

Step 1: $|\hat{B} - B^*|_1$ small implies that $err(\hat{G}, G)$ is small.

New Objective: Show that $\langle X^T X - \hat{\Gamma}, B^* - B \rangle > 0$ as long as $|B^* - B|_1$ is not small

$$\langle X^T X - \hat{\Gamma}, B^* - B \rangle = \langle A\mu\mu^T A^T, B^* - B \rangle + \langle E^T E - \Gamma, B^* - B \rangle + \langle \Gamma - \hat{\Gamma}, B^* - B \rangle + \langle A\mu E^T + E\mu A^T, B^* - B \rangle$$

We focus on the two first terms

Signal Term: $\langle A\mu\mu^T A^T, B^* - B \rangle = \frac{1}{4} \Delta^2 |B^* - B^* B|_1$
Control of the quadratic term: $\langle E^T E - \Gamma, B^* - B \rangle$

$B^*$ is projection operator that averages over element of the same group.

$\leadsto$ Decomposition of $E^T E - \Gamma$ by applying $B^*$ or $(I - B^*)$.
Control of the quadratic term: $\langle \mathbf{E}^T \mathbf{E} - \Gamma, \mathbf{B}^* - \mathbf{B} \rangle$

$\mathbf{B}^*$ is projection operator that averages over element of the same group.
$\rightsquigarrow$ Decomposition of $\mathbf{E}^T \mathbf{E} - \Gamma$ by applying $\mathbf{B}^*$ or $(\mathbf{I} - \mathbf{B}^*)$.

**Step 3**: Control of the Projection Along $\text{Im}(\mathbf{B}^*)$

$$\langle (\mathbf{I} - \mathbf{B}^*)(\mathbf{E}^T \mathbf{E} - \Gamma)(\mathbf{I} - \mathbf{B}^*), \mathbf{B}^* - \mathbf{B} \rangle \leq \|\mathbf{E}^T \mathbf{E} - \Gamma\|_{op} \| (\mathbf{I} - \mathbf{B}^*)[\mathbf{B}^* - \mathbf{B}](\mathbf{I} - \mathbf{B}^*) \|_*$$

$$= \|\mathbf{E}^T \mathbf{E} - \Gamma\|_{op} \frac{1}{2m} |\mathbf{B}^* - \mathbf{B}^* \mathbf{B}|_1$$

$\rightsquigarrow$ *Concentration of $\mathbf{E}^T \mathbf{E}$ in operator norm*
Control of the quadratic term: \( \langle E^T E - \Gamma, B^* - B \rangle \)

\( B^* \) is projection operator that averages over element of the same group. 
\( \rightsquigarrow \) Decomposition of \( E^T E - \Gamma \) by applying \( B^* \) or \( (I - B^*) \). 

**Step 3:** Control of the Projection Along \( \text{Im}(B^*) \)

\[
\langle (I - B^*)(E^T E - \Gamma)(I - B^*), B^* - B \rangle \leq \|E^T E - \Gamma\|_{op} \| (I - B^*)[B^* - B](I - B^*) \|_{\ast} \\
= \|E^T E - \Gamma\|_{op} \frac{1}{2m} |B^* - B^*B|_1
\]

\( \rightsquigarrow \) Concentration of \( E^T E \) in operator norm

**Step 4:** Control of \( \langle B^*(E^T E - \Gamma), B^* - B \rangle \).

*First try:* \( \langle A, B \rangle \leq |A|_\infty |B|_1 \) does not lead to exponential bounds.

*A Second try (Fei and Chen('17)):* \( \langle A, B \rangle \leq \sum_{i=1}^{\|B\|_1} A_{(i)} \), where \( A_{(1)} \geq A_{(2)} \geq \ldots \)

Control of the order statistics \( B^*(E^T E - \Gamma) \) by Hanson-Wright inequality + Union bound
Model 2: graph clustering

(conditional) SBM

Assume that the graph is generated by a SBM with $Q_{jk} =$ probability of connection between nodes of groups $j$ and $k$.

Let $X =$ adjacency matrix of the graph $\in \{0, 1\}^{n \times n}$.

For $a \in G_k^*$: $X_a = [QA]_k: - Q_{kk} e_a + E_a$, where $E_a = X_a - \mathbb{E}[X_a]$

$$\Delta^2 = \min_{j \neq k} \|[QA]_k: - [QA]_j:\|^2 \geq m \times \min_{j \neq k} \|Q_k: - Q_j:\|^2 \geq 2m \lambda_{\min}(Q)^2$$
Partial recovery for SBM

\[ \Delta^2 = \min_{j \neq k} \| [QA]_k: - [QA]_j: \|^2 \geq m \times \min_{j \neq k} \| Q_k: - Q_j: \|^2 \left( \geq 2m \lambda_{\min}(Q)^2 \right) \]

\[ L \geq \| Q \|_{op} \lor 1/m \]

**Theorem (Giraud and V. (’18))**

We set \( s^2 = \Delta^2 / L \). If \( s^2 \gtrsim n/m \) we have \( \mathbb{P}[\text{err}(G, \hat{G}) > e^{-cs^2}] \lesssim 1/n^2 \)
Partial recovery for SBM

\[ \Delta^2 = \min_{j \neq k} \| [QA]_k : - [QA]_j : \|^2 \geq m \times \min_{j \neq k} \| Q_k : - Q_j : \|^2 \quad (\geq 2m \lambda_{\min}(Q)^2) \]

\[ L \geq \| Q \|_{op} \lor 1/m \]

**Theorem (Giraud and V.('18))**

We set \( s^2 = \Delta^2 / L \). If \( s^2 \gtrsim n/m \) we have \( \mathbb{P}[\text{err}(G, \hat{G}) > e^{-cs^2}] \lesssim 1/n^2 \)

if we have enforced \( \| B \|_{op} \leq \frac{K^3}{n} e^{4nL} \).
Assortative case: \( Q = (p - q)I + q11^T \) and \( m = n/K \)

1. \( s^2 = 2m(p - q)^2/p \) for \( p \geq K/n \).

   - Tight constants in Gao et al. ('17), Yun and Proutière ('14)

2. Perfect recovery for

   \[
   \frac{(p - q)^2}{p} \gtrsim \frac{K^2 \vee K \log(n)}{n}
   \]

   Matches best known polynomial time algorithm condition Chen and Xu ('16)
Exponential decay: Abbe and Sandon (’15) consider the scaling $Q = Q_0 \log(n)/n$ for a fixed $K$. Results not completely comparable.

Perfect recovery: if $\|Q\|_{op} = O(\min_{j,k} Q_{j,k})$, we recover (up to constant) the optimal condition of Abbe and Sandon (’15).
Exponential decay: Abbe and Sandon (’15) consider the scaling $Q = Q_0 \log(n)/n$ for a fixed $K$. Results not completely comparable.

Perfect recovery: if $\|Q\|_{op} = O(\min_{j,k} Q_{j,k})$, we recover (up to constant) the optimal condition of Abbe and Sandon (’15)

Other SDP for SBM: Relaxed $K$-means differs from Chen & Xu (’16), Hajek et al. (’16), Guédon & Vershynin (’16), Perry & Wein (’16)...

$$\tilde{B} = \arg\max_{B \in C'} \langle X, B \rangle$$

for assortative graphs ($\text{diag}(Q) > \text{nondiag}(Q)$)
Proof

Same arguments, but:

- spectral control requires trimming arguments in the proof
- control of quadratic terms quite messy due to the symmetry of $X$ (peeling, conditionning, ...)
**Main message**

A corrected convex relaxation of $K$-means achieves some rate-optimal performances in various settings including (conditional) mixture of sub-Gaussian and (conditional) Stochastic Block Model.

Only tuning Parameter is $K$

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C. Giraud and N.V. **Partial recovery bounds for clustering with the relaxed $K$-means.** *Mathematical Statistics and Learning* ArXiv:1807.07547
A corrected convex relaxation of $K$-means achieves some rate-optimal performances in various settings including (conditional) mixture of sub-Gaussian and (conditional) Stochastic Block Model.

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Merci pour votre attention!


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